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MA 495/595
Spring 2024

Homework 6

Due February 22nd on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake. There are some hints on the second page.

1. Borthwick Exercises 6.3, 6.4,
2. Let a and b be real numbers, and let

$$f_n(x) = \frac{1}{n^a + n^b x^2}.$$

   (a) Find the values of a and b for which $f_n(x) \to 0$ in $L^1(\mathbb{R})$ as $n \to \infty$.

   (b) Find the values of a and b for which $f_n(x) \to 0$ in $L^2(\mathbb{R})$ as $n \to \infty$.

   (c) Find the values of a and b for which $f_n(x) \to 0$ in $L^\infty(\mathbb{R})$ as $n \to \infty$.

   (d) Sketch the regions in the (a, b) plane which correspond to the values of a and b found above.

   (e) This is not to hand in, but you may enjoy plotting some of these sequences in Desmos, especially if you find L^p spaces mysterious. Note that in parts (a) and (b) the sequence can have a growing spike near $x = 0$, while in (c) it cannot. Also, if you make a Venn diagram like this `https://static.dexform.com/media/docs/4436/venn-diagram-sample_bg1.png`, where one circle is 'converges in L^1', one circle is 'converges in L^2', and one circle is 'converges in L^∞', then out of the eight regions of the Venn diagram, from the sketch in part (d) you can see that six have possible values of a and b but two have none.

Hints:

1. 6.3 and 6.4 are similar to Theorem 4.12. 6.4b is similar to Corollary 4.13.
2. For (a), you must check whether $\int_{-\infty}^{\infty} |f_n(x)|dx \to 0$. For (b), you must check whether $\int_{-\infty}^{\infty} |f_n(x)|^2(x) \to 0$. For (c), you must check whether $\sup\{|f_n(x)| : x \in \mathbb{R}\} = \max\{|f_n(x)| : x \in \mathbb{R}\} \to 0$. To simplify the integrals, use a substitution of the form $x = n^c y$, where c is chosen in such a way that all of the n dependence comes out of the integral.